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## Alternative $N = (4, 0)$ Superstring and $\sigma$ -Models<sup>1</sup>

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### Abstract

We present an alternative  $N = (4, 0)$  superstring theory, with field content different from that of previously-known  $N = (4, 0)$  superstring theories. This theory is presented as a non-linear  $\sigma$ -model on the coset  $SU(n, 1)/SU(n) \otimes U(1)$  as the target space-time with torsion, which is coupled to  $N = (4, 0)$  world-sheet superconformal gravity. Our result indicates that the target space-time for  $N = 4$  superstring theory does not necessarily have to be a hyper-Kähler or quaternionic Kähler manifold.

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## 1. Introduction

The importance of superstring theories with extended supersymmetries manifests itself in various contexts. One exciting observation [1][2] is that  $N = 2$  superstring [3] has self-dual (supersymmetric) Yang-Mills theory or self-dual (super)gravity theories as its consistent backgrounds [4] in its critical four-dimensional target space-time, which play crucial roles as the possible master theory for integrable systems in lower-dimensions [5][6]. According to a recent analysis [7], the critical dimension of  $N = 4$  superstring theory [8] is probably<sup>3</sup>  $D_c = +4$  instead of  $D_c = -8$  which was the previous common understanding [9]. If this is indeed the case, the  $N = 4$  superstring theory will gain more reason to be regarded as an important theory like the  $N = 2$  superstring theory.

There has been recently [10] some indication that the  $N = 4$  superstring theory is not unique, but there are many different versions, depending on the representation of the matter multiplet. One of the reasons is due to some ambiguity related to what is called mirror transformation that replaces chiral superfields by twisted chiral superfields [10].

Independent of this indication, there has been some development about new sets of matter multiplets with global  $N = 8$  supersymmetry [11], based on what is called dimensional “oxidation” from one dimension ( $D = 1$ ) to two-dimensions ( $D = 2$ ), which is a reversed procedure of the traditional dimensional “reduction”. We can easily apply this oxidation technique to get unknown matter multiplets also for the  $N = 4$  supersymmetry [11].

In this paper, we present an alternative  $N = (4, 0)$ <sup>4</sup> matter scalar multiplet which can be coupled to  $N = (4, 0)$  superconformal gravity with a new field content obtained by the oxidation technique [11] different from the previously-known superstring theories [12][13]. In particular, we promote it to a heterotic non-linear  $\sigma$ -model on a Kähler manifold [14], when coupling to the  $N = (4, 0)$  supergravity. Our results indicate that the previously-known  $N = (4, 0)$  theories [12][13] are not the only valid ones as acceptable superstring theories.

We first list up our possible matter scalar multiplets, which are called SM-I through SM-IV with global  $N = (4, 0)$  supersymmetry obtained by the dimensional oxidation. We next couple the SM-I to  $N = (4, 0)$  superconformal gravity, with all the scalars parametrizing the coordinates of the coset  $SU(n, 1)/SU(n) \otimes U(1)$ .

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<sup>3</sup>The dimensionalities here are counted in terms of bosonic scalar fields.

<sup>4</sup>In our paper we present  $N = (p, q)$  as  $p$  supersymmetries with positive chirality and  $q$  supersymmetries with negative chirality.

## 2. Global $N = (4, 0)$ Scalar Multiplets

We first review the possible global  $N = (4, 0)$  scalar multiplets [11], that are obtained by the oxidation of  $D = 1$  theories, also for notational clarification. There are in total four scalar multiplets, and we call them SM-I through SM-IV [11]. All of these multiplets have  $4 + 4$  on-shell degrees of freedom. We give below their transformation rules and corresponding invariant lagrangians:

(i) SM-I  $(\mathcal{A}, \mathcal{A}^*, \mathcal{B}, \mathcal{B}^*, \psi^A)$ :<sup>5</sup>

$$\begin{aligned}\delta_Q \mathcal{A} &= +2(\epsilon^A \psi_A) \equiv +2(\epsilon \psi) \quad , \quad \delta_Q \mathcal{B} = +2i(\bar{\epsilon}^A \psi_A) \equiv +2i(\bar{\epsilon} \psi) \quad , \\ \delta_Q \psi^A &= -i\gamma^\mu \bar{\epsilon}^A \partial_\mu \mathcal{A} + \gamma^\mu \epsilon^A \partial_\mu \mathcal{B} \quad , \\ \mathcal{L}_I &= +|\partial_\mu \mathcal{A}|^2 + |\partial_\mu \mathcal{B}|^2 - 2i(\psi^A \gamma^\mu \partial_\mu \bar{\psi}_A) \quad .\end{aligned}\tag{2.1}$$

The indices  $A, B, \dots = 1, 2$  are for the **2**-representation of the intrinsic  $SU(2)$  group of the  $N = (4, 0)$  superconformal gravity. The  $\mathcal{A}$  and  $\mathcal{B}$ -fields are complex scalars, and  $\psi^A$  are Weyl fermions with negative chirality, once we chose the Weyl fermionic parameter  $\epsilon^A$  to have positive chirality for the unidexterous  $N = (4, 0)$  supersymmetry.<sup>6</sup> The superscript  $*$  denotes its complex-conjugate. In this paper, we usually omit the fermionic chirality indices  $+$ ,  $-$ , unless needed for explaining complex-conjugations, but it is useful to keep in mind that all the gravitini carry the positive chirality. Other conventions such as the indices  $\mu, \nu, \dots = 0, 1$  for the world-sheet curved coordinates are self-explanatory.

(ii) SM-II  $(\phi, \phi^I, \lambda_A, \bar{\lambda}^A)$ :

$$\begin{aligned}\delta_Q \phi &= (\epsilon^A \lambda_A) - (\bar{\epsilon}^A \bar{\lambda}_A) \equiv (\epsilon \lambda) - (\bar{\epsilon} \bar{\lambda}) \quad , \\ \delta_Q \phi^I &= -2(\epsilon T^I \lambda) + 2(\bar{\epsilon} T^I \bar{\lambda}) \quad , \\ \delta_Q \lambda_A &= -i\gamma^\mu \bar{\epsilon}_A \partial_\mu \phi - 2i\gamma^\mu (T^I \bar{\epsilon})_A \partial_\mu \phi^I \quad , \\ \mathcal{L}_{II} &= +\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(\partial_\mu \phi^I)^2 + i(\bar{\lambda}^A \gamma^\mu \partial_\mu \lambda_A) \quad .\end{aligned}\tag{2.2}$$

The indices  $I, J, \dots = 1, 2, 3$  are for the adjoint representation for the  $SU(2)$ , and  $T^I$ 's are its generators related to the Pauli's  $\sigma$ -matrices by  $T^I = -(i/2)\sigma^I$ , so that  $T^I T^J = -(1/4)\delta^{IJ} + (1/2)\epsilon^{IJK}T^K$ . The action of  $T^I$  is such as  $(T^I \bar{\epsilon})_A = (T^I)_A{}^B \bar{\epsilon}_B = -(T^I)_{AB} \bar{\epsilon}^B$ . The scalars  $\phi$  and  $\phi^I$  are real, while the fermions  $\lambda_A$  are Weyl with negative chirality.

<sup>5</sup>Our notation in this paper is  $\{\gamma_m, \gamma_n\} = 2\eta_{mn} = 2\text{diag.}(-, +)$ ,  $\epsilon_{01} = +1$ ,  $\gamma_3 = \gamma_0 \gamma_1$ ,  $\gamma^0 = -i\sigma^2$ ,  $\gamma^1 = \sigma^1$ , so  $\gamma^m \gamma^n = \eta^{mn} + \epsilon^{mn} \gamma_3$ . We omit the indices  $+$ ,  $-$  for fermionic chiralities to save space in this paper.

<sup>6</sup>Note that the charge conjugation matrix changes the chiralities: *e.g.*,  $\psi_{+A} = \psi_{-A}^+ C_{+-}$ ,  $\epsilon_{-}^A = \epsilon_{+}^A C_{+-}$ .

(iii) SM-III  $(A_A, A^{*A}, \rho, \bar{\rho}, \pi, \bar{\pi})$ :

$$\begin{aligned}
\delta_Q A_A &= -(\epsilon_A \pi) + (\bar{\epsilon}_A \rho) \quad , \\
\delta_Q \rho &= -2i\gamma^\mu \epsilon^A \partial_\mu A_A \quad , \\
\delta_Q \pi &= -2i\gamma^\mu \bar{\epsilon}^A \partial_\mu A_A \quad , \\
\mathcal{L}_{\text{III}} &= +(\partial_\mu A^{*A})(\partial^\mu A_A) + \frac{i}{2}\bar{\rho}\gamma^\mu \partial_\mu \rho + \frac{i}{2}(\bar{\pi}\gamma^\mu \partial_\mu \pi) \quad .
\end{aligned} \tag{2.3}$$

The  $\rho$  and  $\pi$  are both Weyl fermions with negative chirality, while each component of  $A_A$  is a complex scalar.

(iv) SM-IV  $(B_A, B^{*A}, \psi, \psi_A^B)$ :

$$\begin{aligned}
\delta_Q B_A &= (\bar{\epsilon}_A \psi) - 2i(\bar{\epsilon}_B \psi_A^B) \quad , \\
\delta_Q \psi &= -i\gamma^\mu \epsilon^A \partial_\mu B_A - i\gamma^\mu \bar{\epsilon}_A \partial_\mu B^{*A} \quad , \\
\delta_Q \psi_A^B &= \gamma^\mu \epsilon^B \partial_\mu B_A - \gamma^\mu \bar{\epsilon}_A \partial_\mu B^{*B} - \frac{1}{2}\delta_A^B (\gamma^\mu \epsilon^C \partial_\mu B_C - \gamma^\mu \bar{\epsilon}^C \partial_\mu B_C^*) \quad , \\
\mathcal{L}_{\text{IV}} &= +(\partial_\mu B^{*A})(\partial^\mu B_A) + \frac{i}{2}(\psi\gamma^\mu \partial_\mu \psi) + i(\psi_A^B \gamma^\mu \partial_\mu \psi_B^A) \quad .
\end{aligned} \tag{2.4}$$

Each of  $B_A$  is a complex scalar, while  $\psi$  and  $\psi_A^B$  are Majorana-Weyl fermions with negative chirality. The  $A^B$ -indices on  $\psi_A^B$  denote the  $\mathbf{3}$ -representation of  $SU(2)$ .

### 3. SM-I Coupled to $N = (4, 0)$ Superconformal Gravity

Before considering  $\sigma$ -model couplings, we first fix the notations for the  $N = (4, 0)$  superconformal gravity with the field content  $(e_\mu^m, \psi_\mu^{+A}, \bar{\psi}_\mu^{+A}, B_\mu^I)$ , which is compatible with the field representations of the scalar multiplets SM-I through IV. The transformation rule for the superconformal gravity multiplet is

$$\begin{aligned}
\delta e_\mu^m &= -2i(\epsilon\gamma^m \bar{\psi}_\mu) + 2i(\bar{\epsilon}\gamma^m \psi_\mu) - \Lambda_D e_\mu^m + \epsilon^{mn} \Lambda_M e_{\mu n} \quad , \\
\delta \psi_\mu^A &= \partial_\mu \epsilon^A + \frac{1}{2}\omega_\mu \epsilon^A + B_\mu^I (T^I \epsilon)^A + i\gamma_\mu \eta^A - \frac{1}{2}\Lambda_D \psi_\mu^A - \frac{1}{2}\Lambda_M \psi_\mu^A - \Lambda^I (T^I \psi_\mu)^A \\
&\equiv \mathcal{D}_\mu \epsilon^A + i\gamma_\mu \eta^A - \frac{1}{2}\Lambda_D \psi_\mu^A - \frac{1}{2}\Lambda_M \psi_\mu^A - \Lambda^I (T^I \psi_\mu)^A \quad , \\
\delta \bar{\psi}_{\mu A} &= \partial_\mu \bar{\epsilon}_A + \frac{1}{2}\omega_\mu \bar{\epsilon}_A + B_\mu^I (T^I \bar{\epsilon})_A + i\gamma_\mu \bar{\eta}_A - \frac{1}{2}\Lambda_D \bar{\psi}_{\mu A} - \frac{1}{2}\Lambda_M \bar{\psi}_{\mu A} - \Lambda^I (T^I \bar{\psi}_\mu)_A \\
&\equiv \mathcal{D}_\mu \bar{\epsilon}_A + i\gamma_\mu \bar{\eta}_A - \frac{1}{2}\Lambda_D \bar{\psi}_{\mu A} - \frac{1}{2}\Lambda_M \bar{\psi}_{\mu A} - \Lambda^I (T^I \bar{\psi}_\mu)_A \quad , \\
\delta B_\mu^I &= +4i(\bar{\epsilon} T^I \gamma^\nu \mathcal{R}_{\mu\nu}) - 4i(\epsilon T^I \gamma^\nu \bar{\mathcal{R}}_{\mu\nu}) + 4(\bar{\psi}_\nu T^I \gamma_\mu \gamma^\nu \eta) - 4(\psi_\nu T^I \gamma_\mu \gamma^\nu \bar{\eta}) \\
&\quad + \frac{1}{2}(\delta_\mu^\nu - e^{-1} \epsilon_\mu^\nu) \mathcal{D}_\nu \Lambda^I + \frac{1}{2}(\delta_\mu^\nu + e^{-1} \epsilon_\mu^\nu) \mathcal{D}_\nu \Lambda'^I \quad , \\
\delta \omega_\mu &= -2i(\bar{\epsilon} \gamma^\nu \mathcal{R}_{\mu\nu}) + 2i(\epsilon \gamma^\nu \bar{\mathcal{R}}_{\mu\nu}) - 2(\bar{\psi}_\nu \gamma_\mu \gamma^\nu \eta) + 2(\psi_\nu \gamma_\mu \gamma^\nu \bar{\eta}) + \partial_\mu \Lambda_M - e^{-1} \epsilon_\mu^\nu \partial_\nu \Lambda_D \quad .
\end{aligned} \tag{3.1}$$

The gravitino field strengths  $\mathcal{R}_{\mu\nu}{}^A$  and  $\overline{\mathcal{R}}_{\mu\nu A}$  are defined in terms of the covariant derivative  $\mathcal{D}_\mu$  defined in the gravitino  $Q$ -supertranslation. We sometimes omit the contracted  $A, B, \dots$  indices, following the general rule such as  $(\epsilon^A \gamma^m \overline{\psi}_{\mu A}) \equiv +(\epsilon \gamma^m \overline{\psi}_\mu)$ ,  $(\overline{\epsilon}^A \gamma^m \psi_{\mu A}) \equiv +(\overline{\epsilon} \gamma^m \psi_\mu)$ , *etc.* The parameters  $\Lambda_D$ ,  $\Lambda_M$ ,  $\Lambda^I$ ,  $\Lambda'^I$  are respectively the dilatation, Lorentz, the positive and negative chirality parts of the  $SU(2)$  gauge transformations, while  $\omega_\mu \equiv -e^{-1} \epsilon^{\rho\sigma} e_\mu{}^m [\partial_\rho e_{\sigma m} - 2i(\overline{\psi}_\rho \gamma_m \psi_\sigma)]$  is the Lorentz connection.

As is usual with heterotic conformal supergravity [13], the  $Q$ -supertranslation for the gravitino does not have any matter-dependent terms. This is due to the fact that the heterotic supergravity multiplet is already off-shell with enough degrees of freedom.

We now turn to the construction of  $\sigma$ -model couplings of SM-I through SM-IV to the  $N = (4, 0)$  conformal supergravity. We see that the SM-II and SM-IV have similar field contents, in the sense that the representations of the scalars and fermions are flipped between these two multiplets. When we have a scalar field in a multiplet that has non-trivial representation under the intrinsic  $SU(2)$  group of the  $N = (4, 0)$  superconformal gravity, we have to consider its minimal coupling to the  $SU(2)$  gauge field  $B_\mu{}^I$ . However, as a simple trial reveals, such a minimal coupling generates a Brans-Dicke term problematic for conformal symmetry. To be more specific, the variation of the  $SU(2)$  gauge field  $B_\mu{}^I$  in such a minimal coupling generates a term like  $(\epsilon \gamma^\nu \mathcal{R}_{\mu\nu}) \times (\text{scalar}) \partial_\mu (\text{scalar})$  with the gravitino field strength  $\mathcal{R}_{\mu\nu}$  under the  $Q$ -supertranslation of the  $SU(2)$  gauge field. This term may be cancelled by a new term such as  $(\text{fermion}) \gamma^\mu \gamma^\nu \mathcal{R}_{\mu\nu} \times (\text{scalar})$  in the lagrangian. However, the  $Q$ -supertranslation of the gravitino strength of this new term in turn necessitates a Brans-Dicke term  $R \times (\text{scalar})^2$  proportional to the scalar curvature  $R$ . There is also some indication in superspace supporting this statement [15].

The  $\sigma$ -models given in refs. [12][13] are based on the hyper-Kähler or quaternionic Kähler manifold [16]. In the field representation in [13], we notice that the bosons and fermions are unified in the same representation with four degrees of freedom, carrying the curved indices of the target space-time. This indicates that the  $\sigma$ -model in [13] corresponds to the  $\sigma$ -model generalization of both SM-II and SM-IV, in such a way that the coset structure is compatible with  $N = (4, 0)$  supersymmetry.

Among our multiplets above, we see that SM-I has not been presented before, but has scalars in a simple representation. In particular, the scalar fields are singlet under the  $SU(2)$ , so that they have no minimal couplings to  $B_\mu{}^I$ . Therefore we do not encounter the problem mentioned above, and the coupling to  $N = (4, 0)$  superconformal gravity must be easier. As for the remaining SM-III, we see that scalar fields are in non-trivial representations under the  $SU(2)$ , which will cause the problem above. Motivated by this observation, we present in this paper the couplings of the SM-I to the  $N = (4, 0)$  superconformal gravity.

As for a possible  $\sigma$ -model for the SM-I, we naturally expect that its target space-time would be a hyper-Kähler or quaternionic Kähler manifold, according to other  $N = 4$   $\sigma$ -models [12][13]. However, a simple counting of the degrees of freedom of the scalars and fermions reveals that no quaternionic Kähler manifold seems to fit these representations. Another subtlety is the presence of two complex scalars, whose  $Q$ -supertranslations do not seem to be related to each other in a simple way such as complex-conjugates. Based on this observation, we try to loosen the geometric restriction from the quaternionic Kähler manifold to more general Kähler manifolds with suitable representations. The simplest example appears to be  $SU(n, 1)/SU(n) \otimes U(1)$  of  $n$  complex dimensions, namely  $2n$  real dimensions. In fact, we can show below that this assignment indeed works.

We use the following notation for the field content of the multiplet SM-I, parametrizing our Kähler  $\sigma$ -model coset:  $(\phi^\alpha, \bar{\phi}^{\bar{\alpha}}, \varphi^a, \varphi_a^*, \psi^{+aA}, \bar{\psi}^+_{aA})$ , where the indices  $\alpha, \beta, \dots = 1, 2, \dots, n$  ( $n \in \{1, 2, \dots\}$ ) and their complex-conjugates  $\bar{\alpha}, \bar{\beta}, \dots = 1, 2, \dots, n$  are for the curved complex  $n$ -dimensional coordinates in the coset  $SU(n, 1)/SU(n) \otimes U(1)$ , while the indices  $a, b, \dots = 1, 2, \dots, n$  are for the  $\mathbf{n}$ -representations of the  $SU(n)$  isotropy group in the coset, and  $A, B = 1, 2$  are for the  $\mathbf{2}$ -representation of the intrinsic  $SU(2)$  of the  $N = (4, 0)$  superconformal gravity. The  $\phi$  and  $\varphi$ -fields respectively correspond to the previous scalars  $\mathcal{A}$  and  $\mathcal{B}$  in (2.1). In particular, the  $\varphi^a$ -field is identified as a tangent vector under the holonomy group  $SU(n) \otimes U(1)$  in the coset. We sometimes use the indices  $\underline{\alpha} \equiv (\alpha, \bar{\alpha})$ ,  $\underline{\beta} \equiv (\beta, \bar{\beta})$ ,  $\dots$  collectively both for the barred and unbarred curved indices to save space. Since this coset is essentially complex, special treatment is needed for the complex-conjugations of fields. The indices  $a, b, \dots$  as well as  $A, B, \dots$  change their positions from subscript to superscript or *vice versa*, under the complex-conjugations. We will explain more about complex-conjugations shortly.

The vielbein on the coset  $SU(n, 1)/SU(n) \otimes U(1)$  satisfy the usual ortho-normality conditions:

$$\begin{aligned} V_a^\alpha V_\alpha^b &= \delta_a^b, & V_\alpha^a V_a^\beta &= \delta_\alpha^\beta, \\ V^{a\bar{\alpha}} V_{\bar{\alpha}b} &= \delta_b^a, & V_{\bar{\alpha}a} V^{a\bar{\beta}} &= \delta_{\bar{\alpha}}^{\bar{\beta}}. \end{aligned} \quad (3.2)$$

The second line is simply the complex-conjugate of the first one, *i.e.*,  $V^{a\bar{\alpha}} = (V_\alpha^a)^*$  and  $V_{\bar{\alpha}a} = (V_\alpha^a)^*$ . An important fact is that other vielbein components such as  $V_{\alpha a}$  or  $V_a^{\bar{\alpha}}$  simply do *not* exist in our system. This is also related to the fact that we have *no* metric for the  $SU(n)$  indices  $a, b, \dots$ .<sup>7</sup>

Relevantly, our coset has the affinity  $\Gamma_{\underline{\alpha}\underline{\beta}}^\gamma = \left\{ \frac{\gamma}{\underline{\alpha}\underline{\beta}} \right\} + T_{\underline{\alpha}\underline{\beta}}^\gamma$ , where  $T_{\underline{\alpha}\underline{\beta}}^\gamma$  is a totally

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<sup>7</sup>The only exception is the case  $n = 2$ , but even then we can manage all the relevant manipulations without a metric.

antisymmetric torsion tensor whose non-vanishing components are

$$T_{\alpha\beta\bar{\gamma}} = \partial_\alpha B_{\beta\bar{\gamma}} - \partial_\beta B_{\alpha\bar{\gamma}} \quad , \quad (3.3)$$

together with their complex-conjugates, where  $B_{\alpha\bar{\beta}}$  is as usual the antisymmetric tensor in the target space-time that appears in the Wess-Zumino-Witten term in the  $\sigma$ -model. We also notice that the superinvariance of the  $\sigma$ -model action will require the relationship

$$T_{\alpha\beta\bar{\gamma}} = \frac{1}{2} \left( \partial_\beta g_{\alpha\bar{\gamma}} - \partial_\alpha g_{\beta\bar{\gamma}} \right) \quad . \quad (3.4)$$

This feature is similar to the  $N = (2,0)$  heterotic  $\sigma$ -model coupled to superconformal gravity [14][17]. Other useful relations are such as

$$\begin{aligned} \Gamma_{\alpha\bar{\beta}\bar{\gamma}} &= 0 \quad , \quad \Gamma_{\alpha\beta\gamma} = 0 \quad , \\ \Gamma_{\alpha\bar{\beta}\gamma} &= \partial_\alpha g_{\gamma\bar{\beta}} - \partial_\gamma g_{\alpha\bar{\beta}} \quad , \quad \Gamma_{\alpha\beta\bar{\gamma}} = \partial_\beta g_{\alpha\bar{\gamma}} \quad , \end{aligned} \quad (3.5)$$

together with their complex-conjugates. As usual the vielbeins are covariantly constant with respect to the affinity  $\Gamma_{\alpha\bar{\beta}}^{\gamma}$ , and the composite connections  $A_{\alpha a}^b$ ,  $A_{\underline{\alpha}}$  respectively for the  $SU(n)$  and  $U(1)$  holonomy groups in our coset. For instance<sup>8</sup>

$$\mathcal{D}_\alpha V_\beta^c \equiv \partial_\alpha V_\beta^c - \Gamma_{\alpha\beta}^\gamma V_\gamma^c - A_{\alpha c}^b V_\beta^c + i A_\alpha V_\beta^c \equiv 0 \quad . \quad (3.6)$$

Some technical care is needed about the complex-conjugation rules in our system. When taking complex-conjugates, we have to distinguish the basic fields from those produced by the multiplications of  $C_{AB}$  or  $C^{AB}$ . To be more specific, our basic fields are

$$\psi^{aA} \quad , \quad \bar{\psi}_{aA} \quad , \quad \psi_\mu^{+A} \quad , \quad \bar{\psi}_{\mu A} \quad , \quad \epsilon^A \quad , \quad \bar{\epsilon}_A \quad , \quad \eta^A \quad , \quad \bar{\eta}_A \quad , \quad (3.7)$$

while non-basic fields such as  $\psi_{\mu A}$  are regarded as the multiplications of the corresponding basic fields by the  $C_{AB}$  or  $C^{AB}$ :  $\psi_{\mu A} \equiv \psi_\mu^{+B} C_{BA}$ . Illustrative examples are

$$\begin{aligned} (\psi_\mu^{+A})^* &= \bar{\psi}_\mu^{+A} \quad , \quad (\bar{\psi}_\mu^{+A})^* = \psi_\mu^{+A} \quad , \quad (\psi_{\mu-}^A)^* = -\bar{\psi}_{\mu-A} \quad , \quad (\bar{\psi}_{\mu-A})^* = -\psi_{\mu-}^A \quad , \\ (\psi^{-aA})^* &= \bar{\psi}^{-aA} \quad , \quad (\bar{\psi}^{-aA})^* = \psi^{-aA} \quad , \quad (\psi_+^{aA})^* = -\bar{\psi}_{+aA} \quad , \quad (\bar{\psi}_{+aA})^* = -\psi_+^{aA} \quad , \\ (\psi_\mu^{+A})^* &= (\psi_\mu^{+B} C_{BA})^* = (C_{BA})^* (\psi_\mu^{+B})^* = C^{BA} \bar{\psi}_\mu^{+B} = -C^{AB} \bar{\psi}_\mu^{+B} = -\bar{\psi}_\mu^{+A} \quad , \\ (\bar{\psi}_\mu^{+A})^* &= (C^{AB} \bar{\psi}_\mu^{+B})^* = (\bar{\psi}_\mu^{+B})^* (C^{AB})^* = \psi_\mu^{+B} C_{AB} = -\psi_\mu^{+B} C_{BA} = -\psi_\mu^{+A} \quad , \end{aligned} \quad (3.8)$$

obeying also the traditional rules in ref. [18]. Other helpful relations are  $[(\gamma^\mu)_{++}]^* = (\gamma^\mu)_{++}$ ,  $[(\gamma^\mu)_{--}]^* = (\gamma^\mu)_{--}$ ,  $[(T^I)_A{}^B]^* = -(T^I)_B{}^A$ , etc.

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<sup>8</sup>Similar relations hold for the complex-conjugates, and also for *barred* indices  $\bar{\alpha}, \bar{\beta}, \dots$  that we clarify next.

The local transformation rule for the SM-I is generalized from the global case (2.3) as

$$\begin{aligned}
\delta\phi^\alpha &= +2(\epsilon^A\psi^a{}_A)V_a{}^\alpha \quad , \quad \delta\bar\phi^{\bar\alpha} = +2(\bar\epsilon^A\bar\psi_{aA})V^{a\bar\alpha} \quad , \\
\delta\varphi^a &= +2i(\bar\epsilon^A\psi^a{}_A) + (\delta_Q\phi^\alpha)A_{\underline{a}b}{}^a\varphi^b - i(\delta_Q\phi^\alpha)A_{\underline{a}}\varphi^a \quad , \\
\delta\varphi_a^* &= +2i(\epsilon^A\bar\psi_{aA}) - (\delta_Q\phi^\alpha)A_{\underline{a}a}{}^b\varphi_b^* + i(\delta_Q\phi^\alpha)A_{\underline{a}}\varphi_a^* \quad , \\
\delta\psi^{aA} &= -i\gamma^\mu\bar\epsilon^AV_\alpha{}^a\widehat{D}_\mu\phi^\alpha + \gamma^\mu\epsilon^A\widehat{D}_\mu\varphi^a + (\delta_Q\phi^\alpha)A_{\underline{a}b}{}^a\psi^{bA} - i(\delta_Q\phi^\alpha)A_{\underline{a}}\psi^{aA} \\
&\quad + \frac{1}{2}\Lambda_M\psi^{aA} + \frac{1}{2}\Lambda_D\psi^{aA} - \Lambda^I(T^I\psi_a)_A \quad , \\
\delta\bar\psi_{aA} &= +i\gamma^\mu\epsilon_AV_{\bar\alpha}{}^a\widehat{D}_\mu\bar\phi^{\bar\alpha} - \gamma^\mu\bar\epsilon_A\widehat{D}_\mu\varphi_a^* - (\delta_Q\phi^\alpha)A_{\underline{a}a}{}^b\bar\psi_{bA} + i(\delta_Q\phi^\alpha)A_{\underline{a}}\bar\psi_{aA} \\
&\quad + \frac{1}{2}\Lambda_M\bar\psi_{aA} + \frac{1}{2}\Lambda_D\bar\psi_{aA} - \Lambda^I(T^I\bar\psi_a)_A \quad .
\end{aligned} \tag{3.9}$$

As usual, all the covariant derivatives with *hats* are  $Q$ -supercovariant derivatives, *e.g.*,

$$\widehat{D}_\mu\varphi^a \equiv \mathcal{D}_\mu\varphi^a - 2i(\bar\psi_\mu{}^A\psi^a{}_A) \equiv \partial_\mu\varphi^a - (\partial_\mu\phi^\alpha)A_{\underline{a}b}{}^a\varphi^b + i(\partial_\mu\phi^\alpha)A_{\underline{a}}\varphi^a - 2i(\bar\psi_\mu{}^A\psi^a{}_A) \quad , \tag{3.10}$$

where  $\mathcal{D}_\mu\varphi^a$  is the target space-time covariant derivative like (3.6). The only subtlety to be mentioned is the composite connection such as  $(\partial_\mu\phi^\alpha)A_{\underline{a}b}{}^a\varphi^b$ . In the above supercovariantization prescription, the first factor  $(\partial_\mu\phi^\alpha)$  should *not* to be supercovariantized, and this is independent of the statistics of the field that the composite connection is acting on. For more detailed explanation related to quaternionic terms, see *e.g.* refs. [16][19].

We are now ready to present the invariant lagrangian of the heterotic  $\sigma$ -model for the SM-I coupled to the  $N = (4, 0)$  superconformal gravity:

$$\begin{aligned}
e^{-1}\mathcal{L}_I &= +g^{\mu\nu}g_{\alpha\bar\beta}(\partial_\mu\phi^\alpha)(\partial_\nu\bar\phi^{\bar\beta}) + g^{\mu\nu}(\mathcal{D}_\mu\varphi^a)(\mathcal{D}_\nu\varphi_a^*) + \epsilon^{\mu\nu}g_{\alpha\bar\beta}(\partial_\mu\phi^\alpha)(\partial_\nu\bar\phi^{\bar\beta}) \\
&\quad + i(\psi^{aA}\gamma^\mu\mathcal{D}_\mu\bar\psi_{aA}) + i(\bar\psi_{aA}\gamma^\mu\mathcal{D}_\mu\psi^{aA}) \\
&\quad - (\bar\psi_\mu{}^A\gamma^\nu\gamma^\mu\bar\psi_{aA})V_\alpha{}^a\left(\partial_\nu\phi^\alpha + \widehat{D}_\nu\phi^\alpha\right) - (\psi_\mu{}^A\gamma^\nu\gamma^\mu\psi^a{}_A)V_{\bar\alpha}{}^a\left(\partial_\nu\bar\phi^{\bar\alpha} + \widehat{D}_\nu\bar\phi^{\bar\alpha}\right) \\
&\quad - i(\psi_\mu{}^A\gamma^\nu\gamma^\mu\bar\psi_{aA})\left(\mathcal{D}_\nu\varphi^a + \widehat{D}_\nu\varphi^a\right) - i(\bar\psi_\mu{}^A\gamma^\nu\gamma^\mu\psi^a{}_A)\left(\mathcal{D}_\nu\varphi_a^* + \widehat{D}_\nu\varphi_a^*\right) \quad . \tag{3.11}
\end{aligned}$$

The covariant derivative for  $\psi^{aA}$  is defined by

$$\mathcal{D}_\mu\psi^{aA} \equiv \partial_\mu\psi^{aA} - \frac{1}{2}\omega_\mu\psi^{aA} + B_\mu{}^I(T^I\psi^a)^A - (\partial_\mu\phi^\alpha)A_{\underline{a}b}{}^a\psi^{bA} + i(\partial_\mu\phi^\alpha)A_{\underline{a}}\psi^{aA} \quad . \tag{3.12}$$

The superinvariance of the action, including the quartic fermionic terms, can be easily confirmed by inspecting the supercovariance of all the fermionic field equations. It is worthwhile to mention that the the gravitino field equation directly obtained from the lagrangian (3.11) generates the gradient term  $\partial_\mu\epsilon$  under the  $Q$ -supertranslation, indicating its *non*-supercovariance. However, this is superficial and poses no problem, because it turns out to



be proportional to the  $B_\mu{}^I$ -field equation which is nothing but the vanishing  $SU(2)$  current itself.

The absence of purely matter fermionic quartic terms in the lagrangian is a natural feature of our  $N = (4, 0)$  heterotic  $\sigma$ -model. This is because all the matter fermions  $\rho^{-a}$  or  $\pi^{-a}$  carry the same negative chirality, considering also the identity  $(\gamma_m)_{--}(\gamma^m)_{--} \equiv 0$ , it is impossible to form a Lorentz invariant term out of purely matter fermions.

Note that the heterotic  $\sigma$ -model we have presented here shares the same feature with  $N = 2$  heterotic  $\sigma$ -models [14] *via* the structure of the torsion tensor (3.3) or (3.4) as well as with other  $N = (4, 0)$  heterotic  $\sigma$ -models with quaternionic Kähler structure [16]. The limit to the flat target space-time is smooth by making the  $SU(n)$  curvature vanish, because there is no restriction on the curvature tensor, unlike the quaternionic Kähler manifold in the conventional  $N = 4$   $\sigma$ -models [12][13].

As usual in heterotic systems [13][17], we can add some Majorana-Weyl unidexterous fermion (UF) denoted by  $\chi^{(r)}$  with the positive chirality to the  $\sigma$ -model system above. Its invariant lagrangian is

$$e^{-1}\mathcal{L}_{\text{UF}} = +\frac{i}{2}(\chi^{(r)}\gamma^\mu\mathcal{D}_\mu\chi^{(r)}) + \frac{1}{4}(\chi^{(r)}\gamma^\mu\chi^{(s)})[2(\psi^{aA}\gamma_\mu\bar{\psi}_{bA}) + i(\varphi^a\partial_\mu\varphi_b^* - \varphi_b^*\partial_\mu\varphi^a)]F_a{}^{b(r)(s)} \\ - \frac{i}{4}(\chi^{(r)}\gamma^m\chi^{(s)})\left[(\psi^\alpha\gamma_m\psi^\beta)\varphi^{*\bar{\gamma}}\mathcal{D}_\alpha F_{\beta\bar{\gamma}} + (\bar{\psi}^{\bar{\alpha}}\gamma_m\bar{\psi}^{\bar{\beta}})\varphi^\gamma\mathcal{D}_{\bar{\alpha}}F_{\gamma\bar{\beta}}\right], \quad (3.13)$$

up to sixth-order terms in fundamental fields. The indices  $(r), (s), \dots$  are for any arbitrary representation of the target space-time Yang-Mills group, distinguished from the local Lorentz indices  $m, n, \dots$ , and  $\psi^\alpha \equiv \psi^a V_a{}^\alpha$ ,  $\varphi^\alpha \equiv \varphi^a V_a{}^\alpha$ . The covariant derivative for  $\chi^{(r)}$  is

$$\mathcal{D}_\mu\chi^{(r)} \equiv \partial_\mu\psi^{(r)} + \frac{1}{2}\omega_\mu\chi^{(r)} + (\partial_\mu\phi^\alpha)A_\alpha{}^{(r)(s)}\chi^{(s)} + (\partial_\mu\bar{\phi}^{\bar{\alpha}})A_{\bar{\alpha}}{}^{(r)(s)}\chi^{(s)}. \quad (3.14)$$

The  $A_{\underline{\alpha}}{}^{(r)(s)}$  are the Yang-Mills composite gauge field in the target space-time whose field strength is  $F_{\underline{\alpha}\underline{\beta}}{}^{(r)(s)}$ , while  $F_a{}^{b(r)(s)} \equiv F_{\alpha\beta}{}^{(r)(s)}V_a{}^\alpha V^{b\beta}$ . The transformation rule for  $\chi^{(r)}$  is

$$\delta\chi^{(r)} \equiv -(\delta_Q\phi^\alpha)A_{\underline{\alpha}}{}^{(r)(s)}\chi^{(s)} + \frac{1}{2}[(\delta_Q\varphi^a)\varphi_b^* - (\delta_Q\varphi_b^*)\varphi^a]F_a{}^{b(r)(s)}\chi^{(s)} + \frac{1}{2}\Lambda_D\chi^{(r)} - \frac{1}{2}\Lambda_M\chi^{(r)}, \quad (3.15)$$

up to quartic order terms in fundamental fields.

As usual [20], the superinvariance of  $\mathcal{L}_{\text{UF}}$  requires that

$$F_{\alpha\beta}{}^{(r)(s)} = F_{\bar{\alpha}\bar{\beta}}{}^{(r)(s)} = 0. \quad (3.16)$$

We have confirmed the superinvariance of the action (3.13) up to the sixth-order terms in the fundamental fields multiplied by  $F_{\alpha\beta}{}^{(r)(s)}$  or its covariant derivatives. In the invariance check, it is convenient to go to a special frame where all the composite connection (gauge)

fields vanish, except for their curvatures (field strengths). For example, by adding a composite gauge transformation to (3.15), we can delete completely the r.h.s. of  $\delta_Q \chi^{(r)}$ . We use this convenient frame which is analogous to the “geodesic frame” used in general relativity, in order to simplify drastically all the computations. In this frame, all the transformation rules (3.9) and (3.15) contain only the composite curvatures with *no* bare composite connection fields, and therefore all the terms with these bare composite fields after the variation become unimportant. Since  $\varphi^a$  is dimensionless, its higher-order terms may arise at any order. However, we are confident about the perturbative superinvariance at higher orders in the basic fields.

Even though the heterotic  $\sigma$ -model  $\mathcal{L}_1$  has a structure different from the already-known theories, the UF piece (3.13) has the same feature as the known ones [13] as well as differences. In particular, the target space-time Yang-Mills fields has the same couplings to the UF, while the bosonic terms  $\varphi \partial \varphi^* - \varphi^* \partial \varphi$  also enters into this coupling accompanying the usual  $(\psi \gamma \bar{\psi})$  to form the  $SU(n)$  current combination. Relevantly, the term  $[(\delta_Q \varphi) \varphi^* - (\delta_Q \varphi^*) \varphi] F \chi$  is needed in (3.15). In fact, it is this term that guarantees the supercovariance of the  $\chi$ -field equation, which would have generated the gradient term  $\partial_\mu \epsilon$  in its  $Q$ -supertranslation, if this extra term were absent in  $\delta_Q \chi$ .

#### 4. Concluding Remarks

Based on the new matter scalar multiplets obtained out of  $D = 1$  by the dimensional oxidation technique [11], we have presented an alternative  $N = 4$  superstring  $\sigma$ -model with field contents different from any previously-known  $N = 4$  superstring theories. We have established convenient categorization of possible scalar multiplets which may couple to superconformal gravity. There are four possible  $N = 4$  matter scalar multiplets, and two of them are promoted to the previously-known  $\sigma$ -models in refs. [12][13], when coupled to superconformal gravity.

We have seen that our Kähler  $\sigma$ -model based on the SM-I has a mixed property of  $N = (2, 0)$  heterotic  $\sigma$ -models [14][17] and other previous  $N = (4, 0)$  heterotic  $\sigma$ -models [12][13]. The possibility of the non-vanishing torsion tensor, which is proportional to the curl of the antisymmetric tensor in the target space-time, is like the  $N = (2, 0)$  [14][17] or  $N = (4, 0)$  [13] heterotic  $\sigma$ -models.

Another interesting aspects we found in our investigation is that when we couple global matter multiplets to superconformal gravity, the presence of minimal couplings between the  $SU(2)$  gauge field in the supergravity multiplet and matter scalars in non-trivial  $SU(2)$  representations seems to cause troubles for superconformal invariance. In terms of the four scalar

matter multiplets SM-I through SM-IV, we have seen that the SM-II and SM-IV correspond to the heterotic  $\sigma$ -model in ref. [13], while the SM-I is promoted to our new heterotic  $\sigma$ -model on a Kähler manifold. Since all the scalars in the SM-III are non-singlets under the  $SU(2)$ , this multiplet appears to have the problem with coupling to superconformal gravity.

We stress the importance of our results even for the superconformal gravity (3.1) itself, with the gravitini in the simple complex **2**-representation of the  $SU(2)$  group, which has not been presented before.<sup>9</sup> This particular representation was motivated by the matter multiplets SM-I through SM-IV obtained by the dimensional oxidation. Furthermore, the scalar representation in SM-I motivates the study of the complex coset Kähler manifold  $SU(n,1)/SU(n) \otimes U(1)$  as the appropriate  $\sigma$ -model among other potentially possible Kähler manifolds for  $N = (4,0)$  superstring models.

It is also to be emphasized that one of the two scalars in the global SM-I multiplet (2.1) is promoted to be the coordinates of this coset, while the other one plays a role of a tangent vector under the holonomy group  $SU(n) \otimes U(1)$ . This is a new peculiar feature discovered in the study of our  $N = (4,0)$  heterotic  $\sigma$ -model, which has not been presented in the past to our knowledge. The explicit technical treatment of complex coordinates for the Kähler manifold given in this paper has clarified the subtlety related to complex-conjugations, which will be also useful in the future. Another interesting feature of our model is the peculiar UF lagrangian (3.13), where the  $SU(n)$  current couples to the target space-time Yang-Mills field strength.

Even though we chose in this paper the representation  $\psi^{aA}$  for the fermion, it is to be straightforward to rewrite the whole system in terms of  $\psi^{\alpha A}$  with the curved target space-time curved index  $\alpha$ . In such a case, the system will share more common technical features with the model in ref. [13].

According to the conventional wisdom [12][13], only those hyper-Kähler manifolds, that can be promoted to be quaternionic Kähler manifolds, are supposed to be the suitable cosets for  $N = (4,0)$  heterotic  $\sigma$ -models. In this paper, we have established a completely new possibility of  $N = (4,0)$  heterotic  $\sigma$ -models that requires such simple cosets as  $SU(n,1)/SU(n) \otimes U(1)$ . This coset is just a simple example of other possible cosets, which are compatible with our complex-representations in SM-I. Especially, we found that the dimensionality of the target space-time for the  $N = 4$  superstring has to be no longer a multiple of four in real coordinates, but it can be an even integer. This coincides with the intuitive understanding that the number of scalar fields parametrizing the target manifold in our  $N = (4,0)$  system is half of the conventional one [12][13], leaving the remaining

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<sup>9</sup>For example in [12] the gravitini carry the same **2**-indices, but they also have additional  $Sp(1)$  indices subject to what is called symplectic Majorana condition.

super-partner scalar fields living on its tangent space. It is also very suggestive that the recently discovered discrete symmetry  $SL(2, \mathbb{Z}) \subset SL(2, \mathbb{R})$  called  $S$ -duality [21] is also related to our coset *via*  $SL(2, \mathbb{R})/SO(2) \approx SU(1, 1)/U(1)$ , because  $N = 4$  superstring theory may well be the fundamental underlying theory for ordinary  $N = 1$  superstring theory, just like the  $N = 2$  case [6].

The purpose of investigating heterotic  $\sigma$ -models coupled to superconformal gravity is to seek the consistent realization of  $N = 4$  superconformal algebra with explicit representations of matter multiplets. It is these explicit representations that enable us to promote the superconformal algebra to a string theory. In fact, even though superconformal algebra exists for *arbitrary*  $N$  up to  $N = \infty$  [17], there is such restriction as  $N \leq 4$  for realizing superstring theories due to the consistency of the matter representation coupled to superconformal gravity. It seems that a superconformal algebra *without* matter representations is *not* explicit enough to construct a string theory.

As was already mentioned, the possible critical dimension  $D_c = +4$  [7] provides the strong motivation for the intensive study of the  $N = 4$  superstring, not only for usual applications to high-energy particle physics, but also as an important superconformal theory for lower-dimensional supersymmetric integrable systems [6].

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